Lecture 1

Dynamics in Minkowski spacetime

Plan for the lecture

- System of units

- Particle mechanics in Minkowski spacetime

- Electrodynamics in relativistic notation

System of units

Main physical quantities: length, Time, mass, Temperature

SI units: m s kg K

Fundamental constants of Nature:

- Speed of light $c = 3\times 10^{8}\ m/s$
- Planck constant $\hbar = \frac{h}{2\pi} = 1.05\times 10^{-34}\ kg\, m^2/s$
- Newton constant $G = 6.67\times 10^{-11}\ m^3\, kg^{-1}\, s^{-2}$
- Boltzman constant $k_B = 1.38\times 10^{-23}\ m^2\, kg\, s^{-2}\, K^{-1}$

Why are the fundamental constants so small/large in SI units?

→ Hierarchy problem

System of units

Main physical quantities : length , Time , mass , Temperature

SI units : m s Kg K

Planck units : $L_p = \sqrt{\frac{\hbar G}{c^3}} = 1$, $t_p = \frac{L_p}{c} = 1$, $M_p = \sqrt{\frac{\hbar c}{G}} = 1$, $\frac{M_p c^2}{k_B} = 1$

$\Downarrow$

simpler formulas

Fundamental constants of Nature :

- Speed of light $c = 3 \times 10^{8}\ m/s = 1$
- Planck constant $\hbar = \frac{h}{2\pi} = 1.05 \times 10^{-34}\ kg\, m^2/s = 1$
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System of units

Main physical quantities : length , Time , mass , Temperature

SI units : m s Kg K

Planck units : 1 1 1 1

HEP units : GeV^{-1} GeV^{-1} GeV GeV

High Energy Physics

Fundamental constants of Nature :

- Speed of light $c = 3\times 10^{8}\, m/s = 1 = 1$
- Planck constant $\hbar = \dfrac{h}{2\pi} = 1.05\times 10^{-34}\, Kg\, m^2/s = 1 = 1$
- Newton constant $G = 6.67\times 10^{-11}\, m^3\, Kg^{-1}\, s^{-2} = 1 = M_p^{-2} = \left[1.22\times 10^{19}\, GeV\right]^{-2}$
- Boltzman constant $k_B = 1.38\times 10^{-23}\, m^2\, Kg\, s^{-2}\, K^{-1} = 1 = 1$

System of units

Main physical quantities : length , time , mass , Temperature

	length	time	mass	Temperature
SI units :	m	s	Kg	K
Planck units :	1	1	1	1
HEP units :	GeV^{-1}	GeV^{-1}	GeV	GeV

GeV ← 10^9 electron . Volt

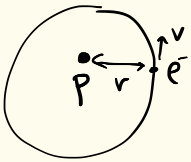
High Energy Physics (HEP) units are well adapted to particle physics :

proton's mass $\simeq 0.938$ GeV

proton radius $\simeq 4.3\ \mathrm{GeV}^{-1}$

$\Rightarrow$ Powerful physical estimates

Example: Atoms



$$E = m_e v^2 - \alpha \frac{1}{r} \sim \overbrace{m_e v^2}^{K} - \overbrace{\alpha m_e v}^{V}$$

$$K \sim V \Rightarrow v \sim \alpha = \frac{1}{137}$$

$$r \sim \frac{1}{m_e \alpha}$$

$$L = m_e v r \sim 1$$

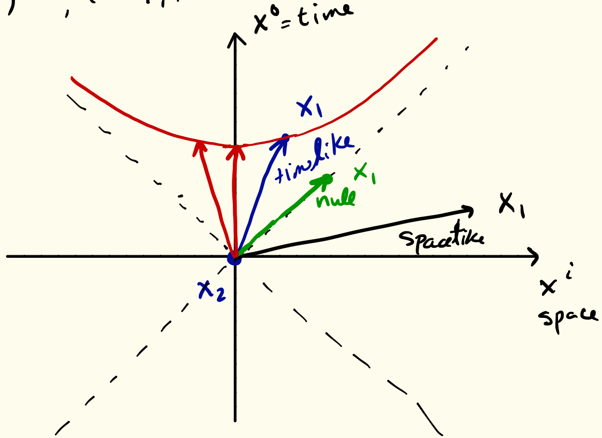
$$E \sim m_e \alpha^2 = m_n \underbrace{\frac{m_e}{m_n}}_{10^{-7}} \alpha^2$$

$$r \sim \underbrace{\frac{1}{m_n}}_{\mathrm{GeV}^{-1}} \underbrace{\frac{m_n}{m_e}}_{10^3} \underbrace{\frac{1}{\alpha}}_{10^2}$$

Minkowski SpaceTime

4-vector x^μ , $\mu \in \{0,1,2,3\}$

$$\begin{cases} x^0 = \overset{\overset{1}{\parallel}}{c}\, t \\ x^i = (\vec{x})^i \end{cases} , \quad i \in \{1,2,3\}$$



Interval : $\Delta^\mu_{12} = x^\mu_1 - x^\mu_2$

$$\Delta^2_{12} = \eta_{\mu\nu} \Delta^\mu_{12} \Delta^\nu_{12} = -(t_1 - t_2)^2 + (\vec{x}_1 - \vec{x}_2)^2$$

$\eta_{\mu\nu}$: Minkowski metric

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Delta^2_{12} < 0$: Δ_{12} is a time-like vector

$\Delta^2_{12} = 0$: " " " null or lighlike vector

$\Delta^2_{12} > 0$: " " " spacelike vector

Suppose $\Delta^2_{12} < 0$ then there is a frame where $\Delta^\mu_{12} = \left(\pm\sqrt{-\Delta^2_{12}} \,, 0 \right)$

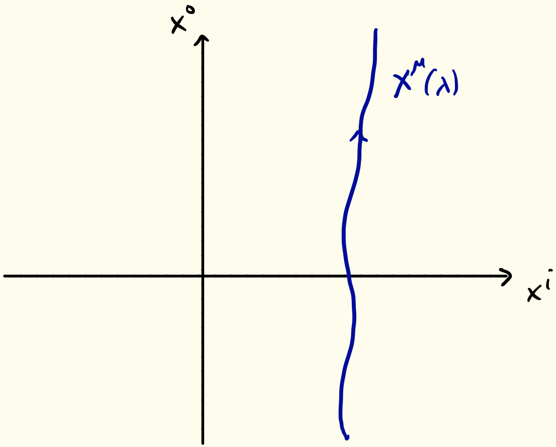
Proper time : $\Delta\tau = \sqrt{-\Delta^2_{12}}$ time in the rest frame

Minkowski SpaceTime

4-vector x^μ , $\mu \in \{0,1,2,3\}$, $x^0 = \overset{\overset{1}{\shortparallel}}{c}t$, $x^i = (\vec{x})^i$, $i \in \{1,2,3\}$

Particle trajectory : $X^\mu(\lambda)$

$\lambda \in \mathbb{R}$
arbitrary parameter



Proper time

$$d\tau^2 = -\eta_{\mu\nu}\, dx^\mu dx^\nu = -\sum_{\mu,\nu=0}^{3} \eta_{\mu\nu}\, dx^\mu dx^\nu$$

Einstein
notation

Action :

$$S = -m \int d\tau = -m \int d\lambda \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$

mass

Symmetries

$$S[x^\mu(\lambda)] = -m \int d\lambda \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$

- **Reparametrization invariance**

$$\lambda = f(\theta) \quad , \quad f'(\theta) > 0$$

$$\tilde{x}^\mu(\theta) = x^\mu(\lambda = f(\theta))$$

Check : $S[\tilde{x}^\mu(\theta)] = S[x^\mu(\lambda)]$

- **Translation invariance**

$$S[x^\mu(\lambda) + a^\mu] = S[x^\mu(\lambda)]$$

a^μ: constant vector

- <u>Lorentz invariance</u>

constant matrix

$$S[\Lambda^\mu{}_\alpha x^\alpha(\lambda)] = S[x^\mu(\lambda)]$$

$$\boxed{\eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}}$$

<u>Lorentz boost</u> (along x^1)

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\gamma = \frac{1}{\sqrt{1-v^2}}$$

<u>Rotation</u>

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 \; 0 \; 0 \\ 0 & \\ 0 & R \\ 0 & \end{pmatrix}$$

$$R \in O(3) \qquad R R^T = \mathbb{1}_{3\times 3}$$

Non - relativistic limit

$$S[x^\mu(\lambda)] = -m \int d\lambda \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$

Choose $\lambda = x^0 = t$

$$S = -m \int dt \sqrt{1 - \vec{v}^2} \quad , \quad \vec{v} = \frac{d\vec{x}}{dt}$$

$|\vec{v}| << c = 1$

$$S = -m \int dt \left[1 - \frac{1}{2}\vec{v}^2 + O(\vec{v}^4) \right] = \int dt \left[-m + \overbrace{\tfrac{1}{2} m \vec{v}^2}^{\text{NR kinetic energy}} + O(\vec{v}^4) \right]$$

$$S_{NR} = \int dt \ \frac{1}{2} m \dot{\vec{x}}^2 \quad \Rightarrow \quad \text{EOM} : \ \ddot{\vec{x}} = 0$$

symmetries :
$$\begin{cases} \text{translations} : & x^\mu \to x^\mu + a^\mu \\ \text{Rotations} : & (\vec{x})^a = R^a{}_b (\vec{x})^b \quad , \quad R R^T = \mathbb{1}_{3\times 3} \\ \text{Galilean boosts} : & \vec{x} \to \vec{x} + \vec{v}_0 t \end{cases}$$

<table>
<tr><th></th><th colspan="3">Symmetries</th></tr>
<tr><td>Newtonian Mechanics
$$m_i \ddot{\vec{x}}_i = -\vec{\nabla}_{x_i} V = -\underbrace{\sum_{j\neq i} \frac{\vec{x}_i - \vec{x}_j}{|\vec{x}_i - \vec{x}_j|}\, U(|\vec{x}_i - \vec{x}_j|)}_{\text{force from 2-body potential } U(|\vec{x}_i - \vec{x}_j|)}$$</td><td rowspan="2">Space and Time Translations
$\vec{x} \to \vec{x} + \vec{a}$
$t \to t + t_0$</td><td rowspan="2">Rotations
$x^a \to R^a{}_b\, x^b$
$R^T R = \mathbb{1}$
$R \in O(3)$
orthogonal matrix</td><td>Galilean Boosts
$\vec{x} \to \vec{x} + \vec{v}\, t$</td></tr>
<tr><td>Minkowski Spacetime</td><td>Lorentz Boosts
different</td></tr>
<tr><td># of parameters =</td><td>4</td><td>3</td><td>3</td></tr>
</table>

$$\text{Lorentz group} = \underset{\substack{\uparrow \\ 1\,\ominus\text{ sign},\ 3\,\oplus\text{ signs}}}{O(1,3)} = \Big\{ \Lambda \in \underbrace{GL(4,\mathbb{R})}_{\text{general linear } 4\times 4} : \Lambda \eta \Lambda^T = \eta \Big\} \supset \begin{cases} \text{Rotations} \\ \text{Lorentz Boosts} \end{cases}$$

$\eta = \mathrm{diag}(-1,1,1,1)$

$$\text{Poincaré group} = \text{Lorentz group} + \text{Translations} \quad \leftarrow \quad 10 \text{ parameters}$$

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu + b^\mu \,, \qquad \Lambda \in O(1,3) \,, \quad b^\mu \in \mathbb{R}^4$$

Charged Particle in external EM field

4-vector potential: $A^\mu = (\Phi, \vec{A})$

$\Rightarrow A_\mu = \eta_{\mu\nu} A^\nu = (-\Phi, \vec{A})$

$$S = -m \int d\tau + q \int A_\mu dx^\mu$$

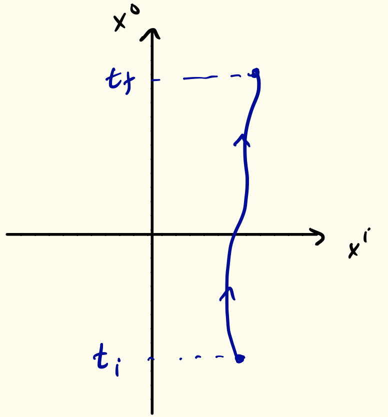
$$S[x(\lambda)] = \int d\lambda \left[-m \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} + q \eta_{\mu\nu} A^\mu(x(\lambda)) \frac{dx^\nu}{d\lambda} \right] \overset{\lambda = t}{=} \int dt \left[-m\sqrt{1-\vec{v}^2} - q\Phi + q\vec{v}\cdot\vec{A} \right]$$

- Reparametrization inv. ✓
- Lorentz inv. $\quad x^\mu = \Lambda^\mu{}_\nu \tilde{x}^\nu$

$$\boxed{A^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\tilde{x})}$$

- Gauge inv. $\quad A_\mu \to A_\mu + \partial_\mu \alpha$

$$S \to S + q \int d\lambda \underbrace{\partial_\mu \alpha \frac{dx^\mu}{d\lambda}}_{\frac{d\alpha(x^\mu(\lambda))}{d\lambda}} = S + q\left[\alpha(x_f, t_f) - \alpha(x_i, t_i)\right]$$



$\Rightarrow$ EOM only depend on local gauge inv. comb.: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Wisdom of the day

You will not be happy when you get what you want.

See www.williambirvine.com for a modern stoic.

Lecture 2

Electrodynamics in Minkowski spacetime

Recap

Where were we?

System of units

Main physical quantities :	length	time	mass	Temperature
SI units :	m	s	Kg	K
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HEP units :	GeV^{-1}	GeV^{-1}	GeV	GeV

High Energy Physics

Fundamental constants of Nature :

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	Symmetries		
	Space and Time Translations	Rotations	Galilean Boosts
Newtonian Mechanics $m_i \ddot{\vec{x}}_i = \underbrace{-\vec{\nabla}_{x_i} V = -\sum_{j \neq i} \frac{\vec{x}_i - \vec{x}_j}{\lvert \vec{x}_i - \vec{x}_j \rvert} U(\lvert \vec{x}_i - \vec{x}_j \rvert)}_{\text{force from 2-body potential } U(\lvert \vec{x}_i - \vec{x}_j \rvert)}$	$\vec{x} \to \vec{x} + \vec{a}$ $t \to t + t_0$	$x^a \to R^a{}_b x^b$ $R^T R = \mathbb{1}$ $R \in O(3)$ orthogonal matrix	$\vec{x} \to \vec{x} + \vec{v} t$
Minkowski Spacetime			Lorentz Boosts
# of parameters =	4	3	3

different

$$\text{Lorentz group} = \underset{\substack{1 \ominus \text{ sign}, \ 3 \oplus \text{ signs}}}{O(1,3)} = \{ \Lambda \in \underbrace{GL(4,\mathbb{R})}_{\text{general linear } 4\times 4} : \Lambda \eta \Lambda^T = \eta \} \supset \begin{cases} \text{Rotations} \\ \text{Lorentz Boosts} \end{cases}$$

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$$\text{Poincaré group} = \text{Lorentz group} + \text{Translations} \leftarrow 10 \text{ parameters}$$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + b^\mu, \quad \Lambda \in O(1,3), \quad b^\mu \in \mathbb{R}^4$$

Charged Particle in external EM field

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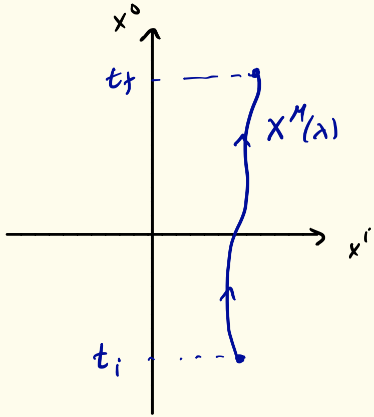
$$S[x(\lambda)] = \int d\lambda \left[-m\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} + q\,\eta_{\mu\nu} A^\mu(x(\lambda))\frac{dx^\nu}{d\lambda}\right] \overset{\lambda = t}{=} \int dt \left[-m\sqrt{1-\vec{v}^2} - q\Phi + q\vec{v}\cdot\vec{A}\right]$$

- Reparametrization inv. ✓
- Lorentz inv. $x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu$ $\qquad \boxed{A^{\mu'}(x') = \Lambda^{\mu'}{}_\nu A^\nu(x)}$
- Gauge inv. $A_\mu \to A_\mu + \partial_\mu \alpha$

$$S \longrightarrow S + q\left[\alpha(x_f, t_f) - \alpha(x_i, t_i)\right]$$



$\Rightarrow$ EOM only depend on local gauge inv. comb.: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

EOM

$$S[x] = \int d\lambda \left[-m\sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} + q A_\mu(x(\lambda))\,\dot{x}^\mu \right] \qquad \bullet \equiv \frac{d}{d\lambda}$$

$$\delta S = S[x+\delta x] - S[x] =$$

$$= \int d\lambda \left[-m\sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - 2\eta_{\mu\nu}\delta\dot{x}^\mu \dot{x}^\nu + O(\delta x^2)} + q A_\mu \delta\dot{x}^\mu + q \overbrace{\begin{bmatrix} \delta x^\nu \partial_\nu A_\mu \\ A_\mu(x) + \end{bmatrix}}^{A_\mu(x+\delta x)} \dot{x}^\mu \right] - S[x]$$

$$= \int d\lambda \left[\underbrace{\frac{m\,\eta_{\mu\nu}\,\delta\dot{x}^\mu \dot{x}^\nu}{\sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}}_{-\delta x_\mu \frac{d}{d\lambda}\left(\frac{m\dot{x}^\mu}{\sqrt{-\dot{x}_\alpha \dot{x}^\alpha}}\right)} + \underbrace{q A_\mu \delta\dot{x}^\mu + q\,\delta x^\nu \partial_\nu A_\mu \dot{x}^\mu}_{\delta x_\mu\left(-q\dot{A}^\mu + q\,\partial^\mu A^\nu \dot{x}_\nu\right)} \right] + O(\delta x^2)$$

$$\underbrace{\dot{x}_\nu \partial^\nu A^\mu}_{q\,\dot{x}_\nu F^{\mu\nu}}$$

$$A_\mu B^\mu = (A^\alpha \eta_{\mu\alpha}) B^\mu = A^\alpha B_\alpha$$

$$\dot{A}^\mu = \frac{d}{d\lambda} A^\mu(x(\lambda))$$

$$\delta S = 0 \Rightarrow \boxed{m\frac{d}{d\lambda}\left(\frac{\dot{x}^\mu}{\sqrt{-\dot{x}_\alpha \dot{x}^\alpha}}\right) = q\,F^{\mu\nu}\dot{x}_\nu}$$

Lorentz force

$$\sqrt{z+t} = \sqrt{z} + \frac{1}{2}\frac{t}{\sqrt{z}} + O(t^2)$$

$$\boxed{m \frac{d}{d\lambda}\left(\frac{\dot{x}^\mu}{\sqrt{-\dot{x}_\alpha \dot{x}^\alpha}}\right) = q F^{\mu\nu} \dot{x}_\nu}$$

$\downarrow$ if $\lambda = \tau$ proper time $\frac{dx^\mu}{d\tau} = U^\mu$, $U^\mu U_\mu = -1$

4-velocity

$$m \frac{d^2 x^\mu}{d\tau^2} = q F^{\mu\nu} \frac{dx_\nu}{d\tau}$$

$$\boxed{m \frac{d}{d\lambda}\left(\frac{\dot{x}^\mu}{\sqrt{-\dot{x}^\alpha \dot{x}_\alpha}}\right) = q\, F^{\mu\nu}\, \dot{x}_\nu}$$

Exercise: Show that choosing $\lambda = x^0 = t$ leads to:

$$\mu = i \quad \Rightarrow \quad \frac{d}{dt}(\underbrace{m\gamma\vec{v}}_{\vec{p}}) = q\left(\vec{E} + \vec{v}\wedge\vec{B}\right)$$

4-momentum

$$p^\mu = m\frac{dx^\mu}{d\tau} = m\gamma(1, \vec{v})$$

$$\mu = 0 \quad \Rightarrow \quad \frac{d}{dt}(\underbrace{m\gamma}_{p^0}) = q\vec{E}\cdot\vec{v}$$

where we used

$$\vec{v} = \frac{d\vec{x}}{dt}, \quad \gamma = \frac{1}{\sqrt{1-v^2}}, \quad A^\mu = (\Phi, \vec{A})$$

$$F^{0i} = -F_{0i} = -\left(\partial_0 A_i - \partial_i A_0\right) = -\left(\partial_t \vec{A}\right)_i - \left(\vec{\nabla}\Phi\right)_i = \left(\vec{E}\right)_i$$

$$F^{ij} = \dots = \epsilon_{ijk}\left(\vec{B}\right)_k$$

Plan for the lecture

- Tensors and Lorentz transformations
- Electrodynamics in relativistic notation
- The energy - momentum tensor

Tensors and Lorentz transformations

$$\Lambda^{\mu'}{}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} \neq \delta^{\mu'}_{\nu}$$

$$x^{\mu} \xrightarrow{L.T.} x^{\mu'} = \Lambda^{\mu'}{}_{\nu}\, x^{\nu}$$

constant matrix such that $\Lambda^{\mu'}{}_{\alpha} \Lambda^{\nu'}{}_{\beta}\, \eta_{\mu'\nu'} = \eta_{\alpha\beta}$

Matrix notation: $\Lambda^{T} \eta \Lambda = \eta \iff \Lambda \eta^{-1} \Lambda^{T} = \eta^{-1} \iff \Lambda^{\alpha'}{}_{\mu}\, \eta^{\mu\nu}\, \Lambda^{\beta'}{}_{\nu} = \eta^{\alpha'\beta'}$

$\Lambda \eta^{-1}$ $\qquad$ $\Lambda^{-1} \eta^{-1}$

$$\eta^{-1} \eta = \mathbb{1} \iff \eta^{\alpha\beta} \eta_{\beta\gamma} = \delta^{\alpha}_{\gamma}$$

Vector field $\qquad J^{\mu}(x) \longrightarrow \boxed{J^{\mu'}(x') = \Lambda^{\mu'}{}_{\nu}\, J^{\nu}(x)}$

Covector field is a linear map on the space of vectors field

$$\omega(J) = \omega_{\mu}(x)\, J^{\mu}(x) \xrightarrow{L.T.} \omega_{\mu'}(x')\, J^{\mu'}(x') = \underbrace{\omega_{\mu'}(x')\, \Lambda^{\mu'}{}_{\nu}}_{\omega_{\mu}(x)}\, J^{\nu}(x)$$

$$\boxed{\Lambda^{\mu'}{}_{\nu}\, \omega_{\mu'}(x') = \omega_{\nu}(x)}$$

Tensors and Lorentz transformations

$$x^\mu \xrightarrow{L.T.} x^{\mu'} = \Lambda^{\mu'}{}_\nu \, x^\nu$$

$$\boxed{J^{\mu'}(x') = \Lambda^{\mu'}{}_\nu \, J^\nu(x)} \qquad \boxed{\Lambda^{\mu'}{}_\nu \, \omega_{\mu'}(x') = \omega_\nu(x)} \; \circledast$$

indices can be lowered or raised with the metric :

example : $$A^\mu \underbrace{B^\nu \eta_{\mu\nu}}_{\equiv B_\mu} \xrightarrow{L.T.} \underbrace{\Lambda^{\mu'}{}_\alpha A^\alpha}_{A^{\mu'}} \underbrace{\Lambda^{\nu'}{}_\beta B^\beta}_{B^{\nu'}} \eta_{\mu'\nu'} = A^\alpha B^\beta \eta_{\alpha\beta}$$

B_μ transforms as a co-vector

$$\underbrace{\eta_{\gamma'\beta'} \underbrace{\Lambda^{\beta'}{}_\alpha \eta^{\alpha\nu} \Lambda^{\mu'}{}_\nu}_{\eta^{\beta'\mu'}} \omega_{\mu'}(x')}_{\omega_{\gamma'}(x')} = \omega_\nu(x) \underbrace{\eta_{\gamma'\mu'} \Lambda^{\beta'}{}_\alpha \eta^{\alpha\nu}}_{\equiv \Lambda_{\gamma'}{}^\nu}$$

$\Lambda_{\gamma'}{}^\nu$ (inverse L.T.)

$\equiv \Lambda^\nu{}_{\gamma'} = \dfrac{dx^\nu}{dx^{\gamma'}}$ (notation of S. Carroll)

Tensors and Lorentz transformations

$$x^\mu \xrightarrow{L.T.} x^{\mu'} = \Lambda^{\mu'}{}_\nu \, x^\nu$$

More general tensor field

$$T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}(x) \to T^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \cdots \nu_\ell'}(x') = \Lambda^{\mu_1'}{}_{\mu_1} \cdots \Lambda^{\mu_k'}{}_{\mu_k} \Lambda_{\nu_1'}{}^{\nu_1} \cdots \Lambda_{\nu_\ell'}{}^{\nu_\ell} \, T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}(x)$$

One can make new tensors using index contractions :

$$T^\nu{}_{\nu\mu} \equiv V_\mu \qquad \text{covector}$$

Electrodynamics

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu \right]$$

current 4-vector

$$J^\mu = (\rho, \vec{J})$$

charge density

current density

$$= \int dt \int d^3x \left[\frac{1}{2} \vec{E}^2 - \frac{1}{2} \vec{B}^2 - \Phi \rho + \vec{A} \cdot \vec{J} \right]$$

EOM:

$$\delta S = S[A_\mu + \delta A_\mu] - S[A_\mu] = \int d^4x \left[-\frac{1}{2} F^{\mu\nu} \overbrace{\delta F_{\mu\nu}}^{2\,\partial_\mu \delta A_\nu \,\cancel{- \partial_\nu \delta A_\mu}} + J^\mu \delta A_\mu \right]$$

$$= \int d^4x \, \delta A_\nu \left[\underbrace{\partial_\mu F^{\mu\nu} + J^\nu}_{= 0} \right]$$

Maxwell equations

$$F^{\mu\nu} T_{\mu\nu} = -F^{\mu\nu} T_{\nu\mu}$$

$$= F^{\nu\mu} T_{\nu\mu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \Longrightarrow \quad \boxed{\partial_{[\alpha} F_{\mu\nu]} = 0}$$

antisymmetrize

$$T_{[\alpha_1 \cdots \alpha_n]} \equiv \frac{1}{n!} \sum_{\substack{\text{perm} \\ \sigma}} T_{\sigma(\alpha_1 \cdots \alpha_n)} \, (-1)^{\text{sign}(\sigma)}$$

$$(\quad) \qquad\qquad\qquad\qquad\qquad\qquad\qquad 1$$

$$T_{[\alpha\beta]} = \frac{1}{2}\left(T_{\alpha\beta} - T_{\beta\alpha} \right)$$

$$0 = \partial_\nu \partial_\mu F^{\mu\nu} = -\partial_\nu J^\nu$$

$$S = \int d^4x \left[-\frac{1}{4} F^{\mu\nu} F^{\alpha\beta} \eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\nu} A^\mu J^\nu \right]$$

- Symmetries $\qquad S[A,J] = S[\tilde{A},\tilde{J}]$

  - translations
$$\tilde{A}^\nu(\tilde{x}^\mu) = A^\nu(x^\nu) \qquad \tilde{x}^\mu = x^\mu + a^\mu$$
$$\tilde{J}^\nu(\tilde{x}) = J^\nu(x)$$

  - Lorentz
$$\tilde{A}^\mu(\tilde{x}) = \Lambda^\mu{}_\nu A^\nu(x) \qquad \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu$$
$$\tilde{J}^\mu(\tilde{x}) = \Lambda^\mu{}_\nu J^\nu(x)$$

- Gauge inv. $\qquad A_\mu \to A_\mu + \partial_\mu \alpha$

$$\partial_t \rho + \vec{\nabla} \cdot \vec{J} = 0$$

$$S \to S + \int d^4x \, (\partial_\mu \alpha) J^\mu = S - \int d^4x \, \alpha \, \underbrace{\partial_\mu J^\mu}_{=0}$$

charge conservation

Exercise: Using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with $A^\mu = (\Phi, \vec{A})$ show that

$$\boxed{\partial_\mu F^{\mu\nu} = -J^{\nu}} \quad \Longleftrightarrow \quad \begin{cases} \vec{\nabla}\cdot\vec{E} \cdot \rho \\[2ex] \vec{\nabla}\times\vec{B} = \vec{J} + \dfrac{\partial \vec{E}}{\partial t} \end{cases}$$

$$\boxed{\partial_{[\alpha} F_{\mu\nu]} = 0} \quad \Longleftrightarrow \quad \begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\[2ex] \vec{\nabla}\times\vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0 \end{cases}$$

Wisdom of the day

Financial Independence

FIRE

4% rule

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